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A VARIATIONAL METHOD FOR FINDING PERIODIC SOLUTIONS OF DIFFEREN--ETC(U)

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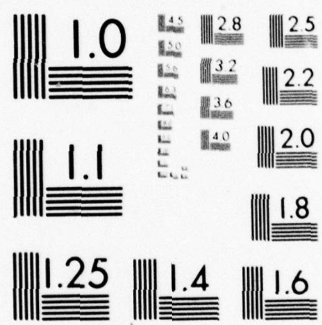

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A VARIATIONAL METHOD FOR FINDING PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper concerns the use of minimax and approximation techniques from the calculus of variations to prove the existence of periodic solutions of Hamiltonian systems of ordinary differential equations. Most of the results are for equations where the period is prescribed and assumptions are made about the growth of the Hamiltonian near infinity. However it is also shown how such results can give information about solutions having prescribed energy.

AMS(MOS) Subject Classifications: 34C15, 34C25.

Key Words: Periodic solution, Hamiltonian system, Energy surface, Semilinear wave equation, Critical point, Variational method, Minimax argument, Index theory.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Hamilton's Principle gives a classical variational characterization of a solution of Hamilton's equations as a critical point of an appropriate functional. We develop a method here which is spiritually related to this principle and which can be used to prove the existence of periodic solutions to Hamilton's equations.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A VARIATIONAL METHOD FOR FINDING PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

Paul H. Rabinowitz

§1. Introduction

Our goal here is to describe a method for finding periodic solutions of ordinary and partial differential equations. More accurately it is a procedure for finding critical points of indefinite functionals. Rather than give an abstract formulation of this method, we prefer to illustrate it in a more concrete setting. Accordingly some applications will be stated followed by their detailed treatment by means of our procedure.

We will mainly stay in the setting of Hamiltonian systems of ordinary differential equations. Thus consider such a system:

$$\dot{p} = -H_q \quad , \quad \dot{q} = H_p \tag{1.1}$$

where $p, q \in \mathbb{R}^n$, $H = H(p,q): \mathbb{R}^{2n} \to \mathbb{R}$, and $\cdot$ denotes d/dt. Equivalently (1.1) can be written as

$$\dot{z} = \mathcal{J} H_z \tag{1.2}$$

where $z = (p,q) \in \mathbb{R}^{2n}$ and $\mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I denoting the identity matrix in $\mathbb{R}^n$.

Our first result concerns the existence of periodic solutions of (1.2) on a prescribed energy surface:

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Theorem 1.3: If $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies

(H_1) $H_z \neq 0$ on $H^{-1}(1)$,

(H_2) $H^{-1}(1)$ is radially diffeomorphic to S^{2n-1}, i.e. the mapping $z \to \frac{z}{|z|}$, $H^{-1}(1) \to S^{2n-1}$ is a diffeomorphism, then (1.2) possesses a periodic solution on $H^{-1}(1)$.

Observe that the period of this solution is a priori unknown and indeed determining it is one of the main difficulties to be overcome in the course of the proof of Theorem 1.3. An interesting open question under the hypotheses of Theorem 1.3 is whether better lower bounds for the number of geometrically distinct solutions can be given. For the special case of $H(z)$ a positive definite quadratic form plus higher order terms, it has been shown by Weinstein [1] that for each small b, (1.2) has at least n distinct periodic orbits on $H^{-1}(b)$. It is tempting to conjecture that the same lower bound holds for our set-up.

Next we state a result for (1.2) where the period rather than the energy is prescribed.

Theorem 1.4: *Suppose* $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ *and satisfies*

(H_3) $H(z) \geq 0$ for all $z \in \mathbb{R}^{2n}$,

(H_4) $H(z) = o(|z|^2)$ at $z = 0$,

(H_5) There is an $\bar{r} > 0$ and $\theta \in (0, \frac{1}{2})$ such that $0 < H(z) \leq \theta(z, H_z(z))_{\mathbb{R}^{2n}}$ for $|z| \geq \bar{r}$.

Then for any $\tau > 0$, (1.2) possesses a nonconstant τ periodic solution.

At first glance, Theorems 1.3 and 1.4 appear to be rather different results, but in fact Theorem 1.3 can be obtained as a simple consequence of Theorem 1.4. Alternatively, a direct proof can be given following the lines of our solution procedure. The ideas that are used in the proof of Theorem 1.4 work equally well if H depends explicitly on t in a time periodic fashion, i.e. we have a forced rather than free vibration situation, and one seeks a solution of (1.2) having the same period as the forcing term.

We suspect that a sharper conclusion obtains under the hypotheses of Theorem 1.4, namely for all $\tau > 0$, (1.2) possesses a nonconstant periodic solution with τ as minimal period. To merely get a τ periodic solution does not require the full strength of the hypotheses of Theorem 1.4. In fact we have the following generalization of this result:

Theorem 1.5: Suppose $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies (H_5). Then for any $\tau, \hat{r} > 0$, there is a τ periodic solution $z(t)$ of (1.2) having $\|z\|_{L^\infty} > \hat{r}$.

Simple examples show the period τ need not be minimal if we only assume (H_5). Theorem 1.4 is of course a consequence of Theorem 1.5. However we prefer to give separate proofs of these results since the latter requires the introduction of some additional topological machinery which can be bypassed in proving Theorem 1.4 directly.

For comparison purposes, we conclude our list of theorems by stating an analogue of Theorem 1.4 for a partial differential equation. Consider the semilinear wave equation

$$(1.6)\quad \begin{cases} u_{tt} - u_{xx} + f(u) = 0 \quad , \quad 0 < x < \pi,\ t \in \mathbb{R} \\ u(0,t) = 0 = u(\pi,t) \ . \end{cases}$$

Theorem 1.7: Suppose $f \in C^2(\mathbb{R}, \mathbb{R})$ and satisfies

(f_1) f is strictly monotone increasing,

(f_2) $f(r) = o(|r|)$ at $r = 0$,

(f_3) there are constants $\bar{r} > 0$ and $\theta \in (0, \frac{1}{2})$ such that $F(r) = \int_0^r f(s)ds \le \theta\, r\, f(r)$ for $r \ge \bar{r}$.

Then for any τ which is a rational multiple of π, (1.6) possesses a nontrivial classical solution which is τ periodic in t.

The greater technicalities involved in working with (1.6) required imposing more restrictions on the nonlinearity f and on the period τ than in Theorem 1.4. We do not know whether (f_1) or the rationality condition on $\tau\pi^{-1}$ can be eliminated. Likewise it is not known if there is an analogue of Theorem 1.3 in this setting. The details of the proof of Theorem 1.7 can be found in [2] and will not be further discussed here.

Our approach towards the above results is by means of the calculus of variations. We try to find solutions of (1.2) or (1.6) as critical points of corresponding functionals. For example, in the context of Theorem 1.4 with $\tau = 2\pi$, we seek critical points of the corresponding Lagrangian:

$$\int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - H(z)]dt \tag{1.8}$$

while for (1.6) (and $\tau = 2\pi$) the analogue of (1.8) is

$$\int_0^{2\pi}\int_0^{\pi} [\tfrac{1}{2}(u_t^2 - u_x^2) - F(u)]dx\, dt \quad . \tag{1.9}$$

To treat the set up of Theorem 1.3, we first make a change of time variable $t \to 2\pi\tau^{-1}t \equiv \lambda^{-1}t$, where τ is the unknown period, so that (1.2) transforms to

$$\dot{z} = \lambda \mathcal{J} H_z \tag{1.10}$$

and the unknown period becomes 2π. Then working in the class of 2π periodic functions, we search for critical points of the action integral

$$A(z) = \int_0^{2\pi} (p,\dot{q})_{\mathbb{R}^n} dt \tag{1.11}$$

subject to the constraint

$$\frac{1}{2\pi}\int_0^{2\pi} H(z)dt = 1 \quad . \tag{1.12}$$

Formally the unknown period then appears in (1.10) via the Lagrange multiplier λ.

As was mentioned at the beginning of this section, the above functionals are indefinite. In particular, they are neither bounded from above nor from below and the quadratic parts of (1.8) and (1.9) have infinite dimensional subspaces on which they are positive and on which they are negative. Thus obtaining critical points of (1.8), (1.9), or (1.11) - (1.12) is a subtle matter and we do not know how to carry this out in any direct fashion. An approximation procedure is used instead. First the functional is restricted to a finite dimensional subspace of $(L^2(S^1))^{2n}$. Secondly a minimax argument is employed to obtain a critical value and corresponding nontrivial critical point for the finite dimensional problem. Thirdly the minimax characterization of the critical value is used to obtain bounds for the critical value and critical point. Having sufficient estimates, we can use standard arguments to pass to a limit to find a solution of (1.2) (or (1.6)). Lastly in the context of Theorems 1.4, 1.5, or 1.7, an additional argument is required to be sure that the solution obtained is nontrivial. We shall give a detailed illustration of this method in §2.

There does not seem to have been much work of the nature of the above theorems in the literature. Our results, Theorems 1.3 - 1.4 can be found in [3]. Theorem 1.5 is new. Earlier Seifert [4] studied the Euler-Lagrange equations corresponding to the Lagrangian $Q - U$ where $Q(x,\dot{x}) = \sum a_{ij}(x)\dot{x}_i\dot{x}_j$ is positive definite in $\dot{x}$, $a_{ij}(x)$ and $U(x)$ are real analytic in a domain $G \subset \mathbb{R}^n$, $U = E$ and $U_x \neq 0$ on ∂G, $U < E$ in G, and G is homeomorphic to the unit ball in $\mathbb{R}^n$. Using geodesic arguments from differential geometry, he showed that the Euler-Lagrange equations for $Q - U$ possess a time periodic solution with energy E. More recently, in work done concurrently with our own, Weinstein [5] extended Seifert's arguments and results replacing $Q - U$ by $H(p,q) = K(p,q) + U(q)$ where U is as above and K is even and convex in p for fixed q. As an application, he obtained a variant of Theorem 1.3 with (H_2) replaced by the condition that $H^{-1}(1)$ bounds a convex region. Some other results of a special nature have been obtained for related problems by Berger [6], Gordon [7], Clark [8], Jacobowitz [9], and Hartman [10]. A considerable amount of work has also been carried out on bifurcation questions for Hamiltonian systems. We refer the reader to Berger [6], Weinstein [1], [11], Moser [12], Bottkol [13], Chow-Mallet-Paret [14], and Fadell-Rabinowitz [15] for more information.

Theorem 1.4 will be proved in §2 using the procedure outlined above. Then an elementary proof of Theorem 1.3 will be carried out in §3 using Theorem 1.4. Lastly in §4 we prove Theorem 1.5. To carry out our method here, we introduce a topological index theory which was developed in [15] and which forms the basis for the minimax constructions used for this theorem.

§2. Proof of Theorem 1.4

The proof of Theorem 1.4 will be given in this section. Observe that no upper bound is placed on the rate of growth of H at infinity. This creates some technical problems which we get around by introducing a new Hamiltonian H_K which coincides with H for $|z| \leq K$ and grows at a prescribed rate at infinity. Let $K > \bar{r}$ and $\chi \in C^{\infty}(\mathbb{R}^+, \mathbb{R}^+)$ such that $\chi(s) = 1$ if $s \leq K$, $\chi(s) = 0$ if $s \geq K + 1$, and $\chi'(s) < 0$ if $s \in (K, K + 1)$. Now set

$$(2.1) \qquad H_K(z) = \chi(|z|)H(z) + (1 - \chi(|z|))\rho|z|^4$$

where $\rho = \rho(K)$ satisfies

$$(2.2) \qquad \rho \geq (K + 1)^{-4} \max_{|z|=K+1} H(z) .$$

Then $H_K \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ and satisfies $(H_3) - (H_4)$. Moreover a calculation using (2.2) and (H_5) shows H_K satisfies (H_5) with θ replaced by $\hat{\theta} = \max(\theta, \frac{1}{4})$. Setting $z = r\,w$ where $w \in S^{2n-1}$, (H_5) implies that

$$(2.3) \qquad \frac{d\, H_K(r\,w)}{dr} \geq \hat{\theta}^{-1}\, r\, H_K(r\,w)$$

for $r > \bar{r}$. On integration (2.3) gives

$$(2.4) \qquad H_K(z) \geq a_1|z|^{\hat{\theta}^{-1}} - a_2$$

for all $z \in \mathbb{R}^{2n}$ where the positive constants a_1, a_2 are independent of K.

The Hamiltonian system corresponding to H_K is

$$(2.5) \qquad \dot{z} = \mathcal{J} H_{Kz} .$$

Instead of seeking τ-periodic solutions of (1.2) or (2.5), it is convenient to make the change of variables $t \to 2\pi\,\tau^{-1}\,t \equiv \lambda^{-1}\,t$ transforming (1.2) and (2.5) into

$$(2.6) \qquad \dot{z} = \lambda\, \mathcal{J} H_z$$

$$(2.7) \qquad \dot{z} = \lambda\, \mathcal{J} H_{Kz}$$

respectively. We seek 2π periodic solutions of (2.6) - (2.7). Theorem 1.4 will be obtained with the aid of the analogous result for (2.7):

Theorem 2.8: If H satisfies (H_3) - (H_5), then for any $K > \bar{r}$ and any $\tau > 0$, (2.7) possesses a nonconstant 2π periodic solution z_K with

$$(2.9) \qquad \int_0^{2\pi} (z_K(t), H_{Kz}(z_K(t)))_{\mathbb{R}^{2n}} dt \leq M_1$$

where M_1 is independent of K.

Proof of Theorem 1.4: For each $K > \bar{r}$, by Theorem 2.8 we have a nonconstant solution z_K of (2.7). It suffices to show that for K sufficiently large, $\|z_K\|_{L^\infty} \leq K$. Then $H_{Kz}(z_K) = H(z_K)$ so z_K satisfies (2.6). The following lemma provides the desired bound on z_K.

Lemma 2.10: There exists a constant M_2 independent of K such that $\|z_K\|_{L^\infty} \leq M_2$.

Proof: By (H_5),

$$(2.11) \qquad H_K(\zeta) \leq \hat{\theta}(\zeta, H_{Kz}(\zeta))_{\mathbb{R}^{2n}} + M_3$$

for all $\zeta \in \mathbb{R}^{2n}$ with M_3 independent of K. Taking $\zeta = z_K(t)$ in (2.11), integrating, and using (2.9) yields

$$(2.12) \qquad \int_0^{2\pi} H_K(z_K) dt \leq \hat{\theta} M_1 + 2\pi M_3 .$$

Since z_K satisfies the Hamiltonian system (2.7), $H_K(z_K)$ is independent of t. Consequently by (2.12),

$$(2.13) \qquad H_K(z_K) \leq \frac{\hat{\theta}}{2\pi} M_1 + M_3 ,$$

and the lemma now follows from (2.4) and (2.13).

The proof of Theorem 2.8 will now be carried out using the program sketched in the Introduction. To begin, set

$$(2.14) \qquad I(z) = \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda H_K(z)] dt$$

where $z(t) = (p(t), q(t))$. Then $I(z)$ is defined on E, the Hilbert space of $2n$ - tuples of 2π periodic functions which are square integrable and have square integrable first derivatives, i.e. $E = (W^{1,2}(S^1))^{2n}$ under the associated inner product. Formally a critical point of I in E is a weak solution of (2.7).

The first step in our solution procedure is to approximate I on E by a finite dimensional problem. This is easily done here. Let e_k, $1 \le k \le 2n$ denote the usual orthonormal bases in $\mathbb{R}^{2n}$, i.e. $e_1 = (1, 0,..,0)$, etc. Set

$$E_m = \text{span}\{(\sin jt)e_k, (\cos jt)e_k \mid 0 \le j \le m, 1 \le k \le 2n\}$$

Now we simply consider I restricted to E_m.

The next step in our program is to obtain a nontrivial critical point for $I|_{E_m}$. The following lemma supplies an existence tool. Let $B_r = \{\xi \in \mathbb{R}^j \mid |\xi| < r\}$. For $k < j$, let $\mathbb{R}^k = \{\xi \in \mathbb{R}^j \mid \xi = (\xi_1,\cdots,\xi_k, 0,\cdots,0\}$ and $(\mathbb{R}^k)^\perp = \{\xi \in \mathbb{R}^j \mid \xi = (0,\cdots,0,\xi_{k+1},\cdots,\xi_j\}$.

Lemma 2.15: Let $\Phi \in C^1(\mathbb{R}^j,\mathbb{R})$, $k < j$, and $\Psi:\mathbb{R}^j \to \mathbb{R}$ such that $\Phi(\xi) \le \Psi(\xi)$ for all $\xi \in \mathbb{R}^j$. Suppose

(Φ_1) $\Psi \le 0$ for all $\xi \in \mathbb{R}^k$

(Φ_2) There is a constant $\delta > 0$ such that $\Phi > 0$ in $(B_\delta\backslash\{0\}) \cap (\mathbb{R}^k)^\perp$

(Φ_3) There is a constant $\mu > 0$ such that $\Psi < 0$ in $\mathbb{R}^j \backslash B_\mu$.

Then Φ has a positive critical value b characterized by

$$b = \inf_{h\in\Gamma} \max_{\xi\in\bar{B}_\mu\cap\mathbb{R}^{k+1}} \Phi(h(\xi)) \tag{2.16}$$

where

$$\Gamma = \{h \in C(\bar{B}_\mu \cap \mathbb{R}^{k+1},\mathbb{R}^j) \mid h(\xi) = \xi \text{ if } \Psi(\xi) \le 0\} .$$

Proof: A proof of Lemma 2.15 can be found in [2] or [16].

To apply Lemma 2.15 to $I|_{E_m}$, identify E_m (under $\|\cdot\|_{L^2}$) with $\mathbb{R}^j$ where $j = 2n(2m + 1)$ and take $\Phi = \Psi = I|_{E_m}$. To verify the hypotheses of the lemma, we introduce a convenient bases in E_m. Set

$$\varphi_{jk} = (\sin jt)e_k - (\cos jt)e_{k+n}, 0 \le j \le m, \ 1 \le k \le n$$

$$\psi_{jk} = (\cos jt)e_k + (\sin jt)e_{k+n}$$

$$\theta_{jk} = (\sin jt)e_k + (\cos jt)e_{k+n}$$

$$\zeta_{jk} + (\cos jt)e_k - (\sin jt)e_{k+n}$$

and take $E^+ = \text{span}\{\varphi_{jk}, \psi_{jk} | j \in \mathbb{N}, 1 \le k \le n\}$, $E^- = \text{span}\{\theta_{jk}, \zeta_{jk} | j \in \mathbb{N}, 1 \le k \le n\}$, $E_m^{\pm} = E^{\pm} \cap E_m$, and $E^0 = \text{span}\{\varphi_{0k}, \psi_{0k} | 1 \le k \le n\}$. Then E_m^+, E_m^-, E^0 are orthogonal subspaces of E_m whose span is E_m. Let

$$A(z) = \int_0^{2\pi} (p,\dot{q})_{\mathbb{R}^n} \, dt \quad ,$$

the action integral. It is easy to verify that $A > 0$ on $E_m^+\backslash\{0\}$, $A < 0$ on $E_m^-\backslash\{0\}$, and $A = 0$ on E^0. Choosing $\mathbb{R}^k = E^0 \oplus E_m^-$, $(\mathbb{R}^k)^{\perp} = E_m^+$, and $\mathbb{R}^{k+1} = E^0 \oplus E_m^- \oplus \text{span}\{\varphi_{11}\} \equiv V_m$, it now follows from (H_3), (H_4), and (H_5) respectively that (Φ_1), (Φ_2), and (Φ_3) are satisfied. Thus by Lemma 2.15, $I|_{E_m}$ has a positive critical value b_m with corresponding critical point z_m.

The third step in our procedure is to use the minimax characterization of b_m to obtain bounds on b_m and z_m.

<u>Lemma 2.17</u>: There are constants M_4, M_5 independent of m and K and constants M_6, M_7 independent of m such that for all $m > 1$,

(2.18) $$b_m \le M_4$$

(2.19) $$\int_0^{2\pi} (z_m, H_{Kz}(z_m))_{\mathbb{R}^{2n}} \, dt \le M_5$$

(2.20) $$\|z_m\|_{L^4} \le M_6$$

(2.21) $$\|z_m\|_E = (\|\dot{z}_m\|_{L^2}^2 + \|z_m\|_{L^2}^2)^{1/2} \le M_7 \ .$$

Proof: Observe that $h(z) \equiv z \in \Gamma$. Hence by (2.16),

(2.22) $$0 < b_m \le \max_{\bar{B}_\mu \cap V_m} I \le \max_{V_m} I$$

where by (Φ_3) max rather than sup can be used in the right hand inequality. Any function $z \in V_m$ can be expressed as

(2.23) $$z(t) = \|z\|_{L^2}(\zeta(t)\cos\omega + (2\pi)^{-1/2}\varphi_{11}(t)\sin\omega)$$

where $\zeta \in E_m^0 \oplus E^-$, $\|\zeta\|_{L^2} = 1$, and $\omega \in [0,2\pi]$. Choosing $z = \hat{z} \in V_m$ which maximizes $I|_{V_m}$, (2.22) - (2.23) show that

(2.24) $$\lambda \int_0^{2\pi} H_K(\hat{z})\,dt \le \frac{1}{2}\|\hat{z}\|^2_{L^2} .$$

Using (2.4) and the Hölder inequality to estimate the right hand side of (2.24) yields

(2.25) $$a_3\|\hat{z}\|^{\hat{\theta}^{-1}}_{L^2} - a_4 \le \frac{1}{2}\|\hat{z}\|^2_{L^2}$$

for some constants a_3, a_4 independent of m and K. Since $\hat{\theta} < \frac{1}{2}$, (2.25) provides a bound on $\|\hat{z}\|_{L^2}$ independent of m and K, say

$$\|\hat{z}\|_{L^2} \le M_8 .$$

Returning to (2.22), we find

(2.26) $$b_m \le I(\hat{z}) \le \frac{1}{2}M_8^2 \equiv M_4 .$$

To verify (2.19), note first that since $z_m \equiv (p_m, q_m)$ is a critical point of $I|_{E_m}$)

(2.27) $$0 = I'(z_m)\zeta = \int_0^{2\pi}[(p_m,\dot{\psi})_{\mathbb{R}^n} + (\varphi,\dot{q}_m)_{\mathbb{R}^n} - \lambda(\zeta,H_{Kz}(z_m))_{\mathbb{R}^{2n}}]dt$$

for all $\zeta = (\varphi,\psi) \in E_m$ where $I'(\xi)\zeta$ denotes the Frechet derivative of I evaluated at ξ and acting on ζ. Using (2.2), (H_5), and some simple estimates, (2.27) with $\zeta = z_m$ gives

(2.28) $$b_m = I(z_m) - \frac{1}{2}I'(z_m)z_m \ge \alpha\int_0^{2\pi}(z_m,H_{Kz}(z_m))_{\mathbb{R}^{2n}}dt - a_5$$

where $\alpha = \min(\frac{1}{2} - \theta, \frac{1}{4})$ and a_5 is independent of m and K. Thus (2.19) follows from (2.28) and (2.18).

The definition of H_K and (2.19) yield (2.20).

Lastly (2.27) is employed again with $\zeta = \mathcal{J}\dot{z}_m$ to obtain (2.21). By the Schwarz inequality,

$$(2.29) \qquad \|\dot{z}_m\|_{L^2} \le \lambda \|H_{Kz}(z_m)\|_{L^2} \le a_6(1 + \|z_m\|^3_{L^6})$$

where a_6 depends on K but not on m. Hence

$$(2.30) \qquad \|z_m\|_E \le \|z_m\|_{L^2} + a_6(1 + \|z_m\|^3_{L^6}) \quad .$$

The Gagliardo-Nirenberg inequality [17] implies that

$$(2.31) \qquad \|z\|_{L^6} \le a_7 \|z\|_E^{1/9} \; \|z\|_{L^4}^{8/9}$$

for all $z \in E$. Hence combining (2.30) - (2.31) and (2.20) gives (2.21).

The fourth step in our proof is to use these estimates to get a solution of (2.7). Indeed it now follows from (2.21), the Sobolev Imbedding Theorem [17], and (2.27) that a subsequence of z_m converges weakly in E and strongly in L^∞ to a continuous function $z_K \equiv (p_K, q_K)$ satisfying

$$(2.32) \qquad 0 = \int_0^{2\pi} [(p_K, \dot{\psi})_{\mathbb{R}^n} + (\varphi, \dot{q}_K)_{\mathbb{R}^n} - \lambda(\zeta, H_{Kz}(z_K))_{\mathbb{R}^{2n}}]dt$$

for all $\zeta = (\varphi, \psi) \in \bigcup_{m>1} E_m \equiv \tilde{E}$. Thus z_K is a weak solution of (2.7). Since $\tilde{E}$ is dense in E, (2.32) implies (2.7) holds a.e. But since $H_{Kz}(z_K)$ is continuous, so is $\dot{z}_K$ and z_K is a classical solution of (2.7). Note also (2.19) implies that z_K satisfies (2.9).

The final step in the proof of Theorem 2.8 is to show that z_K is not a constant. The convergence already established for z_m implies that $b_m = I(z_m) \to I(z_K) \equiv b_K$ along some subsequence. Since $b_m > 0$, $b_K \ge 0$. If z_K is a constant, by (H_3),

$$I(z_K) = -\lambda\int_0^{2\pi} H_K(z_K)\,dt \le 0$$

so $b_K = 0$. The following lemma shows this is not possible and consequently z_K is nonconstant.

Lemma 2.33: $b_K > 0$.

Proof: A lower bound will be obtained for b_K in terms of a comparison problem. By (H_4) and the definition of H_K, for any $\varepsilon > 0$, there is a constant $A_\varepsilon > 0$ and depending on K such that

$$(2.34) \qquad H_K(z) \le \frac{\varepsilon}{2}|z|^2 + \frac{A_\varepsilon}{4}\sum_{i=1}^{2n} z_i^4 \equiv G(z)$$

for all $z \in \mathbb{R}^{2n}$. Set

$$(2.35) \qquad J(z) = \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda G(z)]\,dt \quad .$$

Then by (2.34)-(2.35), $I(z) \le J(z)$ for all $z \in E$. Taking $\Phi = J|_{E_m}$, and $\Psi = I|_{E_m}$, the form of G implies that hypotheses (Φ_1) and (Φ_3) of Lemma 2.15 are satisfied here. Moreover for e.g. $\varepsilon \le \frac{1}{2}$, the quadratic part of J is positive definite on E_m^+ which implies that $J|_{E_m}$ also satisfies (Φ_2). Hence (2.16) defines a critical value c_m of J such that

$$(2.36) \qquad 0 < c_m \le b_m \quad .$$

If w_m is a critical point of $J|_{E_m}$ corresponding to c_m, then

$$(2.37) \qquad c_m = J(w_m) - \frac{1}{2}J'(w_m)w_m$$

$$= \frac{\lambda}{4}A_\varepsilon\int_0^{2\pi}\left(\sum_{i=1}^{2n} w_{mi}^4\right)dt \quad .$$

The estimates of Lemma 2.17 and convergence arguments following it apply to w_m. Hence along a subsequence, $w_m \to w$ satisfying

$$\dot{w} = \lambda \mathcal{J}\, G_z(w)$$

and $c_m \to J(w) \equiv c \ge 0$.

If $b_K = 0$, by (2.36), $c = 0$, and by (2.37), $w \equiv 0$. Therefore $w_m \to 0$ in L^∞. We will show that in fact there is an $\alpha > 0$ such that $\|w_m\|_{L^\infty} \geq \alpha$. Dropping subscripts, we set $w = \bar{w} + W$ where $\bar{w} \in E^0$ and $W \in E_m^+ \oplus E_m^-$. From (2.27) for J' with $\zeta = \bar{w}$, we have

$$(2.38) \qquad 2\pi(\varepsilon|\bar{w}|^2 + A_\varepsilon \sum_{i=1}^{2n} \bar{w}_i^4) = A_\varepsilon \sum_{i=1}^{2n} \int_0^{2\pi} (\bar{w}_i^3 - w_i^3)\bar{w}_i dt \quad .$$

Hence

$$(2.39) \qquad 2\pi \sum_{i=1}^{2n} \bar{w}_i^4 \leq - \sum_{i=1}^{2n} \int_0^{2\pi} (3\bar{w}_i^2 W_i + 3\bar{w}_i W_i^2 + W_i^3)\bar{w}_i dt$$

which together with the Hölder inequality and some simple estimates leads to

$$(2.40) \qquad |\bar{w}| \leq a_8 \|W\|_{L^\infty} \quad .$$

Another application of (2.27) for J' with $\zeta = \mathcal{J}\dot{W}$ yields

$$(2.41) \qquad \|\dot{W}\|_{L^2}^2 \leq 2\varepsilon \|W\|_{L^2}^2 + a_9 \|w\|_{L^\infty}^6$$

where a_9 depends on ε. Since W has mean value zero, it is easy to show that

$$(2.42) \qquad \|W\|_{L^\infty} \leq (2\pi)^{1/2} \|\dot{W}\|_{L^2} \quad .$$

Combining (2.40) - (2.42) gives

$$(2.43) \qquad \|W\|_{L^\infty}^2 \leq 8\pi^2 \varepsilon \|W\|_{L^\infty}^2 + 2\pi a_9 (1 + a_8)^6 \|W\|_{L^\infty}^6 \quad .$$

Since (2.43) is valid for all $\varepsilon \in (0,\frac{1}{2}]$, we choose $\varepsilon = (16\pi^2)^{-1}$. Then (2.43) provides a positive lower bound for $\|W_m\|_{L^\infty}$ and therefore for $\|w_m\|_{L^\infty}$ independently of m.

This completes the proof of Lemma 2.33 and of Theorem 2.8.

Remark 2.44: If H depends explicitly on t in a time periodic fashion and satisfies (H_3) - (H_5), the argument of Theorem 2.8 gives a nonconstant periodic solution of

$$\dot{z} = \lambda \mathcal{J} H_{Kz}(t,z) \tag{2.45}$$

where $H_K(t,z)$ is defined in a similar fashion to (2.1). However the argument of Lemma 2.10 no longer suffices to eliminate the K dependence and some further hypotheses on H seem necessary. See [3].

§3. Proof of Theorem 1.3

In this section we will give an elementary proof of Theorem 1.3 based on Theorem 1.4. To begin we replace H by a more tractable Hamiltonian. The following lemma provides a class of admissable replacements.

Lemma 3.1: Let $H, \overline{H} \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ with $H^{-1}(1) = \overline{H}^{-1}(1)$ and $H_z, \overline{H}_z \neq 0$ on $H^{-1}(1)$. If $\zeta(t)$ satisfies

$$\dot{\zeta} = \mathcal{J} \overline{H}_z(\zeta) \tag{3.2}$$

and $\zeta(0) \in H^{-1}(1)$, then there is a reparametrization $z(t)$ of $\zeta(t)$ which satisfies (1.2). In particular if $\zeta(t)$ is periodic, so is $z(t)$.

Proof: Since $H^{-1}(1), \overline{H}^{-1}(1)$ are level sets for $H, \overline{H}$ respectively, and $\overline{H}^{-1}(1) = H^{-1}(1)$, $H_z(z) = \nu(z)\overline{H}_z(z)$ for all $z \in H^{-1}(1)$ where $0 < \nu(z) \in C(H^{-1}(1), \mathbb{R})$. Moreover since (3.2) is a Hamiltonian system and $\zeta(0) \in H^{-1}(1)$, $\zeta(t)$ lies on $H^{-1}(1)$ for all $t \in \mathbb{R}$. Setting $z(t) = \zeta(r(t))$ where $r(0) = 0$ and r satisfies

$$\frac{dr}{dt} = \nu(\zeta(r(t))) \ , \tag{3.3}$$

it follows that z satisfies (1.2).

For the periodic case, a bit more care must be exercised since the right hand side of (3.3) is merely continuous and therefore the initial value problem need not have a unique solution. If ζ is T-periodic, let $\overline{t}$ be the first positive value of t such that $r(t) = T$. Replace r by

$s(t) = r(t)$, $t \in [0,\bar{t}]$ and $s(t) = jT + r(t-j\bar{t})$ for $t \in [j\bar{t},(j+1)\bar{t}]$, $j \in \mathbb{Z}$. Then it is easy to verify that $s \in C^1(\mathbb{R},\mathbb{R})$ and $z(t) = \zeta(s(t))$ has period $\bar{t}$.

Proof of Theorem 1.3: It suffices to find a periodic solution of (3.2) for an appropriate choice of $\bar{H}$. As in §2, after a change of time variable, (3.2) becomes

$$\dot{z} = \lambda \mathcal{J} \bar{H}_z \tag{3.4}$$

and we seek a 2π periodic solution of (3.4). To define $\bar{H}$, let $\beta \in C^1(H^{-1}(1), S^{2n-1})$ be the mapping given by (H_2). For each $z \in \mathbb{R}^{2n}\setminus\{0\}$, there is a unique $\alpha(z) \in \mathbb{R}^+$ and $w(z) \in H^{-1}(1)$ such that $z = \alpha w$. Indeed $w(z) = \beta^{-1}(\frac{z}{|z|})$ and $\alpha(z) = |z|\ |w(z)|^{-1}$. Let $\bar{H}(0) = 0$ and $\bar{H}(z) = \alpha(z)^4$, $z \neq 0$. Then $\bar{H} \in C^1(\mathbb{R}^{2n},\mathbb{R})$ and satisfies (H_3) - (H_5). In particular by the homogeneity of $\bar{H}$, $\theta = \frac{1}{4}$ in (H_5) and $\bar{H}_z \neq 0$ if $z \neq 0$. Hence by Theorem 1.4 with $\tau = 2\pi$, (3.4) (with $\lambda = 1$) possesses a nonconstant 2π-periodic solution $u(t)$. Since (3.4) is a Hamiltonian system, $\bar{H}(u(t)) \equiv \rho$, a constant. It need not be the case that $\rho = 1$. However, by the homogeneity of $\bar{H}$, for any $\gamma \neq 0$,

$$(\gamma \dot{u}) = \gamma^{-2} \mathcal{J} \bar{H}_z(\gamma u) \tag{3.5}$$

and

$$\bar{H}(\gamma u) = \gamma^4 \rho \ . \tag{3.6}$$

Choosing $\gamma = \rho^{-1/4}$, $\bar{H}(\gamma u) \equiv 1$ and γu is a 2π periodic solution of (3.4) with $\lambda = \gamma^{-2}$. The proof is complete.

Remark 3.7: Using the proof of Theorem 2.8, it is not difficult to obtain upper and lower bounds for λ and then via Lemma 3.1 for the period of the solution of (1.2).

§4. Proof of Theorem 1.5

We follow the procedure used in §2, modifying it where necessary. In particular by eliminating hypotheses (H_3) - (H_4), Lemma 2.15 which provided the existence basis for Theorem 1.4 is no longer applicable and a replacement is needed. To get one, we exploit a group structure inherent in our problem which has not yet been employed.

Let $z(t) \in E$. We can write

$$z(t) = \sum_{j=-\infty}^{\infty} \alpha_j e^{ijt} \equiv \varphi(e^{it})$$

where $\alpha_j \in \mathbb{C}^n$ and $\alpha_{-j} = \bar{\alpha}_j$. Let $(L_\sigma z)(t) = z(t+\sigma)$ for $\sigma \in [0,2\pi]$. This family of translations induces an S^1 action on E given by $(\omega\varphi)(e^{it}) = \varphi(\omega e^{it})$ for $\omega \in S^1$. We call mappings of E to E which commute with this action or real valued functions on E which are constant along orbits of the action equivariant maps and subsets V of E for which $L_\sigma : V \to V$ for all $\sigma \in [0,2\pi]$ are called invariant. It is easy to verify that E_m, $E_m^{\pm}$, and E^0 are invariant subspaces of E and $I(z)$ as defined in (2.14) is an equivariant map. Note also that E^0 is a fixed point set for $\{L_\sigma | \sigma \in [0,2\pi]\}$ and there are isotropy subgroups of the action of arbitrary order in S^1.

To take advantage of the above S^1 action, we will use a cohomology index theory developed in [15]. Let $\mathcal{E}$ denote the family of invariant subsets of $E\backslash\{0\}$.

Lemma 4.1: There is a mapping $i:\mathcal{E} \to \mathbb{N} \cup \{\infty\}$, i.e. an index theory such that for all $U, V \in \mathcal{E}$,

1° If there is an $f \in C(U,V)$ where f is equivariant, then $i(U) \le i(V)$.

2° $i(U \cup V) \le i(U) + i(V)$

3° If U is closed, then there is a closed invariant neighborhood V of U such that $i(V) = i(U)$.

4° For $z \in E\backslash E^0$, if $S^1 z = \{L_\sigma z | \sigma \in [0,2\pi]\}$, then $i(S^1 z) = 1$.

5° If F is an invariant subspace of $(E^0)^\perp$, the L^2 orthogonal complement of E^0, then $i(F \cap \mathcal{S}) = \frac{1}{2}\dim F$ where $\mathcal{S}$ is the unit sphere in E.

6° If U is contained in a finite dimensional subspace of E, $I(U) < \infty$ if and only if $U \cap E^0 = \phi$.

Proof: The definition of index and proofs of its properties can be found in [13].

One further property of $i(\cdot)$ will be useful later.

<u>Lemma 4.2</u>: If $F \subset E_m$ is an invariant subspace containing E^0, $\dim F \geq 2n(m + k + 1)$, and $U \in \mathcal{E}$ with $U \subset E_m$ and $i(U) \geq n(m - k) + 1$, then $F \cap U \neq \phi$.

Proof: The invariance of F implies the same for $F^\perp \cap E_m$. Suppose $F \cap U = \phi$. Then P_m, the L^2 orthogonal projector of E_m to $F^\perp \cap E_m$, belongs to $C(U,(F^\perp \cap E_m)\backslash\{0\})$ and Pm is equivariant. Hence by 1^0 of Lemma 4.1,

$$(4.3) \qquad i(U) \leq i(P_m(U)) \leq i(U_m) \leq i(\mathcal{S} \cap F^\perp \cap E_m)$$

where U_m denotes the radial projection of $P_m(U)$ to $\mathcal{S} \cap F^\perp \cap E_m$. Since $\dim E_m = 2n(2m + 1)$ and $\dim F \geq 2n(m + k + 1)$, $\dim F^\perp \cap E_m \leq 2n(m - k)$. Therefore by 5^0 of Lemma 4.1,

$$(4.4) \qquad i(\mathcal{S} \cap F^\perp \cap E_m) \leq n(m - k) \quad .$$

But (4.3) - (4.4) are contrary to the hypothesis on $i(U)$. Hence $F \cap U \neq \phi$.

Now we can give a variant of Theorem 2.8.

<u>Theorem 4.5</u>: If H satisfies (H_5), then for any $K > \overline{r}$ and $\tau > 0$, (2.7) possesses a 2π periodic solution.

<u>Remark 4.6</u>: As in Theorem 2.8, it is not just existence but also K independent estimates for the solution that are crucial for the sequel. It is inconvenient to present them at this point and they will be stated in the course of the proof.

The notation of §2 will be used in what follows. As earlier we begin by considering $I|_{E_m}$. With the aid of the above index theory and several ideas from [18], we will obtain a family of critical values of this function. The definition of H_K implies that there are constants M and A_M, the latter depending on K, such that

$$(4.7) \qquad H_K(z) \leq M + A_M|z|^4 \equiv \mathcal{J}(z)$$

for all $z \in \mathbb{R}^{2n}$. Since H_K satisfies (H_5), there is an $\overline{R}$ depending on m and K such that for all $R > \overline{R}$, $I(z) < -2\pi\lambda M$ for $z \in E_m \backslash B_R$. We choose any such R for now and will subject it to one further restriction later. Set

$$V_{mk} = E^0 \oplus E_m^- \oplus \mathrm{span}\{\varphi_{ij},\psi_{ij} \mid 1 \leq i \leq k,\ 1 \leq j \leq n\} \quad .$$

Then V_{mk} is an invariant subspace of E_m. Let

(4.8) $\Gamma_m = \{h \in C(E_m, E_m) \mid h$ is an equivariant homeomorphism of E_m onto E_m and $h(u) = u$ if $I(u) \leq -2\pi\lambda M\}$.

The reason for normalizing h by the $-2\pi\lambda M$ term will become clearer later. Now define

(4.9) $$c_{mk} = \inf_{h\in\Gamma_m} \max_{u\in\bar{B}_R\cap V_{mk}} I(h(u)) \quad 1 \leq k \leq m \ .$$

Lemma 4.10: For any $k \leq m$, c_{mk} is a critical value of $I|_{E_m}$ and

(4.11) $$c_{mk} > -2\pi\lambda M \ .$$

We postpone the proof of Lemma 4.10 for now and complete the

Proof of Theorem 4.5: Since $h(z) \equiv z \in \Gamma_m$, by (4.9), (4.11), and (H_5),

$$-2\pi\lambda M < c_{mk} \leq \max_{z\in V_{mk}} I(z) \ .$$

Replacing (2.23) by

$$z(t) = \|z\|_{L^2}(\zeta(t) \cos \omega + \xi(t) \sin \omega)$$

where ζ and ω are as earlier and $\xi \in$ span $\{\varphi_{ij}, \varphi_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq n\}$ with $\|\xi\|_{L^2} = 1$, the proof of Lemma 2.11 proceeds essentially unchanged with the factor of $\frac{1}{2}$ in (2.24) - (2.26) replaced by k. Thus we obtain estimates for c_{mk} and w_{mk}, a corresponding critical point, which are independent of m:

(4.12) $$c_{mk} \leq M_4$$

(4.13) $$\int_0^{2\pi} (w_{mk}, H_{Kz}(w_{mk}))_{\mathbb{R}^{2n}} dt \leq M_5$$

(4.14) $$\|w_{mk}\|_{L^4} \leq M_6$$

(4.15) $$\|w_{mk}\|_E \leq M_7$$

where $M_4 - M_5$ depend only on k and $M_6 - M_7$ depend on k and K. Now as in §2, a subsequence of w_{mk} converges to a function w_k as $m \to \infty$ and w_k satisfies (2.7).

Thus Theorem 4.5 is proved modulo Lemma 4.10. Once (4.11) has been established, the lemma is a consequence of the following result. For $c \in \mathbb{R}$, let $\mathcal{A}_c = \{z \in E_m \mid \Phi(z) \leq c\}$ and $\mathcal{K}_c = \{z \in E_m \mid \Phi(z) = c$ and $\Phi'(z) = 0\}$.

Lemma 4.16: Suppose $\Phi \in C^2(E_m, \mathbb{R})$ is equivariant, there exists an $R > 0$ such that $\Phi(z) < -2\pi\lambda M$ for $z \in E_m \backslash B_R$, $\bar{\varepsilon} > 0$, $c > -2\pi\lambda M$, and $\mathcal{O}$ is any invariant neighborhood of $\mathcal{K}_c$. Then there is an $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times E_m, E_m)$ such that

1° $\eta(s,\cdot)$ is equivariant for each $s \in [0,1]$

2° $\eta(s,\cdot)$ is a homeomorphism of E_m onto E_m for each $s \in [0,1]$

3° $\eta(s,z) = z$ if $\Phi(z) \notin [c-\bar{\varepsilon}, c+\bar{\varepsilon}]$

4° $\eta(1, \mathcal{A}_{c+\varepsilon} \backslash \mathcal{O}) \subset \mathcal{A}_{c-\varepsilon}$

5° If $\mathcal{K}_c = \phi$, $\eta(1, \mathcal{A}_{c+\varepsilon}) \subset \mathcal{A}_{c-\varepsilon}$.

Proof: With the exception of 1°, the Lemma is a special case of a standard result. Therefore we will only sketch the proof indicating in the process why 1° is satisfied. More details can be found in [19] or [20].

By making $\bar{\varepsilon}$ smaller if necessary, we can assume

$$\bar{\varepsilon} < (c + 2\pi\lambda M)4^{-1} \equiv \mu \quad . \tag{4.17}$$

The assumption on R implies $\Phi^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \subset \bar{B}_R$ which is compact (and therefore Φ trivially satisfies the Palais-Smale condition in $\bar{B}_R$). Choose any $\varepsilon \in (0, \bar{\varepsilon})$. The function η is determined as the solution of an ordinary differential equation:

$$\frac{d\eta}{ds} = V(\eta) \quad , \quad \eta(0,z) = z \tag{4.18}$$

for $z \in E_m$. To define V, let $\tilde{A} = \mathcal{A}_{c-\bar{\varepsilon}} \cup (E_m \backslash \mathcal{A}_{c+\bar{\varepsilon}})$ and $\tilde{B} = \mathcal{A}_{c+\varepsilon} \cap (E_m \backslash \mathcal{A}_{c-\varepsilon})$. Note that these sets are invariant and therefore $g(z) = \|z-\tilde{A}\|_{L^2}(\|z-\tilde{A}\|_{L^2} + \|z-\tilde{B}\|_{L^2})^{-1}$ is an equivariant function where $\|z-\tilde{A}\|_{L^2}$ denotes the distance (in E_m) from z to $\tilde{A}$. Observe that $g \equiv 0$ on $\tilde{A}$ and $g \equiv 1$ on $\tilde{B}$. Similarly for δ suitably small - see [19] or [20] to make this precise - we can define a Lipschitz continuous equivariant function f such that $f \equiv 0$ on $\{z \in E_m | \|z-\mathcal{K}_c\|_{L^2} \leq \frac{\delta}{8}\}$, $f \equiv 1$ on $\{z \in E_m | \|z-\mathcal{K}_c\|_{L^2} \geq \frac{\delta}{4}\}$ and $0 \leq f \leq 1$. Next define $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(s) = 1$ if $s \in [0,1]$ and $\varphi(s) = s^{-1}$ if $s > 1$. Finally set $V(z) = -f(z)g(z)\varphi(\|\Phi'(z)\|_{L^2})\Phi'(z)$ for $z \in E_m$. Then by construction V is uniformly bounded, locally Lipschitz continuous, and equivariant on E_m. It follows that the solution $\eta(s,z)$ of (4.18) exists for all $s \in \mathbb{R}$ and satisfies 1^o. The semigroup property for (4.18) gives 2^o and the definition of g implies 3^o. Lastly 4^o - 5^o follow as in [19] or [20].

Assuming (4.11) for now, we give the

Proof of Lemma 4.10: Assume first that $H \in C^2$ and therefore $I \in C^2(E_m, \mathbb{R})$ and that in (4.8), I is replaced by any function Ψ where $|\Psi(z) - I(z)| \leq \mu$ on E_m, μ being defined in (4.17). If c_{mk} is not a critical value of $I|_{E_m}$, we can invoke Lemma 4.16 with $\bar{\varepsilon} = \mu$, $c = c_{mk}$. and $\Phi = \phi$. Choose $h \in \Gamma_m$ such that

$$(4.19) \qquad \max_{u \in \bar{B}_R \cap V_{mk}} I(h(u)) \leq c_{mk} + \varepsilon .$$

By 1^o - 2^o of Lemma 4.16, $\eta(1,h)$ is an equivariant homeomorphism of E_m onto E_m. Moreover if $\Psi(z) \leq -2\pi\lambda M$, $h(z) = z$ and 3^o of Lemma 4.16 shows $\eta(1,h(z)) = z$ provided that $I(z) \notin [c_{mk} - \bar{\varepsilon}, c_{mk} + \bar{\varepsilon}]$. This is certainly satisfied since by our choice of ε; $\Psi(z) \leq -2\pi\lambda M$ implies that $-2\pi\lambda M - \bar{\varepsilon} \leq I(z) \leq -2\pi\lambda M + \bar{\varepsilon} < c_{mk} - \bar{\varepsilon}$. Hence $\eta(1,h) \in \Gamma_m$. Consequently

$$(4.20) \qquad \max_{u \in \bar{B}_R \cap V_{mk}} I(\eta(1,h(u))) \geq c_{mk} .$$

But (4.19) and 5^o of Lemma 4.16 imply that

$$\max I(\eta(1,h(u))) \leq c_{mk} - \varepsilon$$

contrary to (4.20). Hence c_{mk} is a critical value of $I|_{E_m}$.

Now suppose H is merely C^1. Let H_j denote a sequence of C^2 functions which converge to H on $\bar{B}_{K+1}$ in $\mathbb{R}^{2n}$ uniformly in the C^1 norm. Set

$$H_{Kj}(z) = \chi(|z|)H_j(z) + (1 - \chi(|z|))\rho(K)|z|^4$$

for $z \in \mathbb{R}^{2n}$ and

$$I_j(z) = \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda H_{Kj}(z)]dt$$

for $z \in E$. Then the functions $I_j|_{E_m}$ satisfy the hypotheses of Lemma 4.16 for all j sufficiently large and converge to $I|_{E_m}$ in $\bar{B}_R$ uniformly in C^1. For such large j, define c^j_{mk} by (4.9) with I replaced by I_j but Γ_m depending on I. Then c^j_{mk} exceeds $-2\pi\lambda M$ since H_j is close to H. Hence by the case just treated with $\Psi = I$, c^j_{mk} is a critical value of $I_j|_{E_m}$ with corresponding critical point u^j_{mk}. The definition of R implies $u^j_{mk} \in \bar{B}_R$. Hence the compactness of $\bar{B}_R$ and convergence of $I_j|_{E_m}$ to $I|_{E_m}$ imply that along some subsequence $u^j_{mk} \to w_{mk}$ and $c^j_{mk} = I(u^j_{mk}) \to I(w_{mk})$ with w_{mk} a critical point of $I|_{E_m}$. Moreover $I(w_{mk}) = c_{mk}$ as defined in (4.9).

It remains to prove (4.11). This estimate and more follow from a comparison argument. First we define

$$(4.21) \quad \Gamma^*_{mk} = \{S \in E_m \mid S \text{ is compact, invariant, and } S \cap h(\bar{B}_R \cap V_{mk}) \neq \phi \text{ for all } h \in \Gamma_m\} .$$

<u>Lemma 4.22</u>: $\Gamma^*_{mk} \neq \phi$. Indeed if $S \subset B_R \cap E_m^+$ is compact, invariant, and satisfies $i(S) \geq n(m-k) + 1$, then $S \in \Gamma^*_{mk}$.

Proof: Note first that such sets S exist since $i(\mathcal{S} \cap E_m^+) = mn$ via 5° of Lemma 4.1. Let $h \in \Gamma_m$. Since $h(z) = z$ for $z \notin B_R$, $h^{-1}(S) \subset B_R$. Therefore $S \cap h(\bar{B}_R \cap V_{mk}) \neq \phi$ is equivalent to $h^{-1}(S) \cap V_{mk} \neq \phi$. Since h is a homeomorphism, $i(S) = i(h^{-1}(S))$ by 1° of

Lemma 4.1. Moreover $\dim V_{mk} = 2n(m+k+1)$. Hence by Lemma 4.2, $h^{-1}(S) \cap V_{mk} \neq \phi$ and $S \in \Gamma^*_{mk}$.

Another set of numbers can now be defined as follows:

$$(4.23) \qquad c^*_{mk} = \sup_{S \in \Gamma^*_{mk}} \min_{u \in S} I(u) \qquad k < m \quad .$$

Lemma 4.24: $c^*_{mk} = c_{mk}$

Proof: For each $S \in \Gamma^*_{mk}$ and each $h \in \Gamma_m$, there exists a $\zeta \in S \cap h(\bar{B}_R \cap V_{mk})$. Therefore

$$\max_{h(\bar{B}_R \cap V_{mk})} I \geq I(\zeta) \geq \min_S I$$

from which it follows that $c^*_{mk} \leq c_{mk}$.

To prove equality, observe that for each $h \in \Gamma_m$, there is a $\zeta_h \in \bar{B}_R \cap V_{mk}$ such that

$$I(h(\zeta_h)) = \max_{u \in \bar{B}_R \cap V_{mk}} I \quad (h(u)) \quad .$$

Let $S = \{h(S^1 \zeta_h) h \in \Gamma_m\}$ where the notation of 4° of Lemma 4.1 is being employed. Then by construction, $S \in \Gamma^*_{mk}$ and

$$\min_S I \geq c_{mk}$$

so we have equality.

The definition of c^*_{mk} makes it more amenable to lower bounds than c_{mk}. While it is possible to obtain such bounds directly, it is convenient to introduce one more comparison problem. Recall the definition of $\mathcal{J}(z)$ in (4.7). Set

$$(4.25) \qquad \Phi(z) \equiv \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda \mathcal{J}(z)]dt$$

$$= \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda A_M |z|^4]dt - 2\pi\lambda M \quad .$$

Thus is the origin of the mysterious term $-2\pi\lambda M$ in the definition of Γ_m. Equation (4.7) implies that

$$(4.26) \qquad \Phi(z) \leq I(z)$$

for all $z \in E$. Therefore

(4.27) $$c^*_{mk} \geq b^*_{mk} \equiv \sup_{S \in \Gamma^*_{mk}} \min_{u \in S} \Phi(u) .$$

Thus to prove (4.11), it suffices to find an appropriate lower bound for b^*_{mk}. To do this one final set of preliminaries is needed. Any $z \in E^+$ can be written as

$$z = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \alpha_{ij}\varphi_{ij} + \beta_{ij}\psi_{ij} .$$

Therefore

(4.28) $$A(z) = \frac{\pi}{2} \sum_{j=1}^{n} \sum_{i=1}^{\infty} j(|\alpha_{ij}|^2 + |\beta_{ij}|^2) .$$

It follows that $A(z)^{1/2}$ is a (Hilbert space) norm on E^+. Indeed the closure Y of E^* with respect to $A(z)^{1/2}$ is a subspace of the fractional Sobolev space $(W^{1/2,2}(S^1))^{2n}$.

Lemma 4.29: For all $z \in Y$ and $r \in [2,\infty)$, there is a constant ω_r depending only on r such that

(4.30) $$\|z\|_{L^r} \leq \omega_r A(z)^{1/2} ,$$

i.e. Y is continuously imbedded in $(L^r)^{2n}$. Moreover the imbedding is compact.

Proof: The first assertion is a special case of standard results on Fourier series. See e.g. the main theorem on integrals of fractional order in [21]. To prove the compactness, observe that (4.30) and the Schwarz inequality imply

(4.31) $$\|z\|_{L^r} \leq \omega_r \|z\|_{L^2}^{\frac{1}{r}} A(z)^{\frac{r-1}{2r}} .$$

The standard proof of the Rellich lemma - see e.g. [22, p. 169] - implies that Y is compactly embedded in $(L^2)^{2n}$. Therefore if $z_j \rightharpoonup 0$ in Y ($\rightharpoonup$ denoting weak convergence), $z_j \to 0$ in $\|\cdot\|_{L^2}$ and since z_j is bounded in Y, $z_j \to 0$ in $\|\cdot\|_{L^r}$ via (4.31). Hence the imbedding is compact.

Now let $D_{mk} = \text{span}\ \{\varphi_{ij},\psi_{ij} | k \leq i \leq m,\ 1 \leq j \leq n\}$ and $D_k = \overline{\bigcup_{m \geq k} D_{mk}}$ where the closure is taken in Y. By Lemma 4.29, we have

$$(4.32) \qquad \|z\|_{L^4} \le d_k \, A(z)^{1/2}$$

for all $z \in D_k$ where

$$\omega_4 \ge d_k = \sup\{\|z\|_{L^4} \mid z \in D_k \text{ and } A(z) = 1\} \quad .$$

Moreover by compactness assertion of Lemma 4.29, there is $\zeta_k \in D_k$ such that $A(\zeta_k) = 1$ and $\|\zeta_k\|_{L^4} = d_k > 0$.

Lemma 4.33: $d_k \to 0$ monotonically as $k \to \infty$.

Proof: The definition of d_k implies that $d_{k+1} \le d_k$. The definition of D_k implies $\zeta_k \rightharpoonup 0$ in Y and hence $d_k = \|\zeta_k\|_{L^4} \to 0$ by Lemma 4.29.

The proof of (4.11) is now completed by combining Lemma 4.24, (4.27), and the following

Lemma 4.34: $b^*_{mk} > -2\pi\lambda M$

Proof: Let $S_{mk} = \{z \in D_{mk} \mid A(z) = \rho^2\}$. By (4.25) and (4.32) we have

$$(4.35) \qquad \Phi(z) \ge \rho^2 - \lambda A_M d_k^4 \rho^4 - 2\pi\lambda M$$

for all $z \in S_{mk}$. Choosing $\rho_k = (2\lambda A_M d_k^4)^{-1/2}$ leads to

$$(4.36) \qquad \Phi(z) \ge \frac{1}{2}\rho_k^2 - 2\pi\lambda M.$$

Making $R = R(m,K)$ sufficiently large insures that $S_{mk} \subset B_R$. Since S_{mk} is radially homeomorphic to the unit ball in D_{mk}, $i(S_{mk}) = n(m-k+1) \ge n(m-k) + 1$ by 1° and 5° of Lemma 4.1. Therefore Lemma 4.22 shows $S_{mk} \in \Gamma^*_{mk}$. Lastly (4.27) and (4.36) imply $b_{mk} \ge \frac{1}{2}\rho_k^2 - 2\pi\lambda M$ and the proof is complete.

Now finally we can give the

Proof of Theorem 1.5: Fix k. For this prescribed value of k and all m, by Lemma 4.10, c_{mk} is a critical value of $I|_{E_m}$ with a corresponding critical point w_{mk}. Moreover (4.12) and (4.13) provide estimates for c_{mk} and w_{mk} depending on k but independent of m and K. Hence on passing to a limit in m along an appropriate subsequence we get a solution w_k of (2.7) satisfying

$$I(w_k) \equiv c_k \le M_4 \tag{4.37}$$

$$\int_0^{2\pi} (w_k, H_{Kz}(w_k))_{\mathbb{R}^{2n}}\, dt \le M_5 \ . \tag{4.38}$$

The estimate of Lemma 2.10 then shows $\|w_k\|_{L^\infty} \le M_9$ with M_9 depending on k but not K. Hence choosing $K \ge M_9$, we can assume w_k satisfies (2.6).

To complete the proof, it suffices to show that for k sufficiently large, $\|w_k\|_{L^\infty} > \hat{r}$. If this is not the case, fix K at e.g. $\hat{r}$. By (2.6), for all $k \in \mathbb{N}$ we then have

$$\|\dot{w}_k\|_{L^\infty} \le \lambda \|H_z(w_k)\|_{L^\infty} \tag{4.39}$$

and therefore

$$c_k = I(w_k) \le \overline{M} \tag{4.40}$$

where $\overline{M}$ depends on $\max\{|H(z)|, |H_z(z)| \mid |z| \le \hat{r}\}$ but not on k or K. Since along an appropriate subsequence $c_{mk} \to c_k$, then for any fixed k and large m,

$$c_{mk} \le \overline{M} + 1 \ . \tag{4.41}$$

But by Lemma 4.24, (4.27), and (4.36),

$$c_{mk} \ge \min_{S_{mk}} \Phi \ge \frac{1}{2}\rho_k^2 - 2\pi\lambda M = \frac{1}{4\lambda A_M d_k^4} - 2\pi\lambda M \ . \tag{4.42}$$

Since A_M depends only on K which is fixed and $d_k \to 0$ as $k \to \infty$ by Lemma 4.33, we can violate (4.41) by choosing k large enough in (4.42). This contradiction completes the proof.

Remark 4.43: It is not difficult to show that

$$I(z) = \int_0^{2\pi} [(p,\dot{q})_{\mathbb{R}^n} - \lambda H_K(z)]dt$$

satisfies the Palais-Smale condition in E or in

$(W^{\frac{1}{2},2}(S^1))^{2n}$. This suggests that a direct infinite dimensional minimax characterization of critical values of I may be possible. The difficulty of course lies in finding an appropriate class of sets to work with.

Remark 4.44: (H_5) implies that for each b sufficiently large, $H^{-1}(b)$ is radially homeomorphic to S^{2n-1} and $H_z \neq 0$ on $H^{-1}(b)$. Therefore by Theorem 1.3, there is a periodic solution of (1.2) on this surface. If one could establish better estimates for its period than we have been able to, this approach may provide a simpler proof of Theorem 1.5 than the one just given.

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This paper concerns the use of minimax and approximation techniques from the calculus of variations to prove the existence of periodic solutions of Hamiltonian systems of ordinary differential equations. Most of the results are for equations where the period is prescribed and assumptions are made about the growth of the Hamiltonian near infinity. However it is also shown how such results can give information about solutions having prescribed energy.

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